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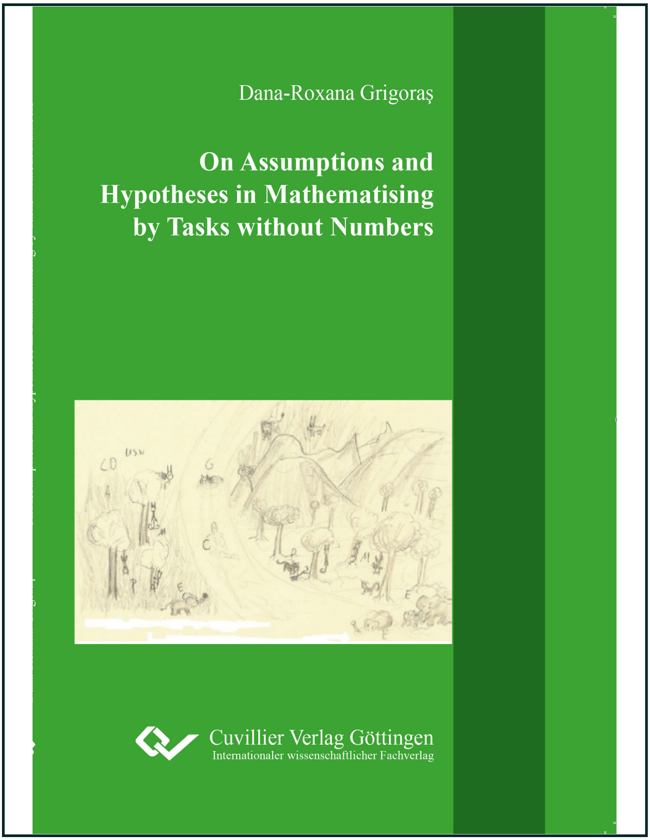
On Assumptions and Hypotheses in Mathematising by Tasks without Numbers



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1 Introduction

This world, and with it human nature, is fated to a never-ending transformation and evolution, in every possible way and by any means. In the course of human development, the main factors that have made human beings the rulers and owners of executive power over all the goods in the world are thinking and soul. Progress is possible only through laborious struggle, and learning is a major component of it. Learning happens in various ways, of which learning from experience is the most basic; and while its roots lie in the very beginning of history, it remains a constant component of the learning process today. Practice shows that if learning takes place in an organised, systematised manner, the chances of better performance are substantially increased. Thinking governs mankind through learning, and therefore much effort has been invested in education as the principal arena in which individuals' mind, character and abilities are formed. As a science of mind, mathematics plays a considerable role and serves as foundation for many other sciences. Hersh states that

> [...] from the viewpoint of philosophy, mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context. [...] call this viewpoint "humanist" (Hersh, 1997, p. xi).

Mathematics – given its complexity as an art of learning – belongs to the leading sciences, at least in the sense of the wide range of its applications. Hence, mathematics education requires special management. In the practice of teaching and learning mathematics, as well as in mathematical research, there exist two categories of stan-dards decreed by the National Council of Teachers of Mathematics (NCTM)[1]. These are *thinking* and *content mathematics*. The four thinking math standards are problem solving, communication, reasoning and connections, while the content mathematics standards are estimation, number sense, geometry and spatial sense, measurement, statistics and probability, fractions and decimals, and patterns and

[1] See also http://www.nctm.org/.

relationships (Fromboluti & Rinck, 1999). In summary, it can be said that mathematics is about cultivating its thinking standards through the acquisition of mathematical contents.

One of the common misconceptions is that numbers are inseparable from mathematics, or the other way around (Zaslavsky, 1994, p. 21). It is also no secret that many school students develop an aversion to mathematics (Zaslavsky, 1994; Furner and Berman, 2003), becoming anxious and incapable of even an average performance when confronted with mathematical problems. It is therefore challenging to know and understand how problems with no apparent mathematical formulation, where no numbers are given and there is nothing to be calculated, are perceived and approached by students. This, in short, is the focus of the present work, in which students' mathematical behaviour, while working on tasks with no obvious mathematical character, is investigated. The main activity that students are involved in is ma-thematising, and this is part of the mathematical modelling.

1.1 Mathematical modelling

The existing literature is rich in material about mathematical modelling. A clarification of this locution may be necessary: how does the word "mathematical" give an attribute to "modelling", which in Webster's dictionary is defined as "a representation of something (usually on a smaller scale)"? Is it to be understood as "modelling by mathematical means" or "modelling with(in) mathematics"? Are these two clearly different meanings? Instead of trying a syntactical analysis, we will mainly consider the first meaning, since it is closer to didactical purposes, while the second sense is not completely separate from the first one, but may be a part of it.

Modelling appears in De Lange's (2003, p. 77) list of competences needed for mathematical literacy and, therefore, competences needed for mathematics as it should be taught. These are defined as

> structuring the field to be modeled; translating reality into mathematical structures; interpreting mathematical models in terms of context or reality; working with models; validating models; reflecting, analyzing, and offering critiques of models or solutions; reflecting on the modelling process (De Lange, 2003, p. 77).

The wide field of mathematical modelling in education is the realm in which many of the actual issues concerning the way to better learning in mathematics continually arise. Over the past thirty years, modelling has featured among the central topics in mathematics education. Above all, teachers' main task is to provide evidence to students that mathematics is not an artificial world, consisting just of numbers, formulas, fields and the like. Students want to understand and see that mathematics is a real tool for solving a multitude of real problems. The connection between mathematics and reality must therefore remain visible for them. However

> [i]t can be strikingly difficult for teachers to change instructional practices in order to emphasize mathematical sense-making, at least in part because many teachers' self-efficacy is based in traditional forms of instruction (Lloyd, 2005, p. 442).

The terminology pertaining to mathematical modelling, applications and matters related to it are pretty abundant. Without claiming to be exhaustive, some clarifications of a few notions and basic concepts used are given below.

In accordance with the meaning adopted by the International Community for the Teaching of Mathematical Modelling (ICTMA)[2], *applications* and *mathematical modelling* are terms that usually come together and denote relationships between mathematics and the real world. By reality, or real world, is meant everything that has to do with everyday life, nature, society and the world around us, including school subjects or scientific disciplines other than mathematics. *Applications* focus on the direction from mathematics to reality, and can be summarised under the ques-

[2] See also http://www.ictma.net/

tion: “Where can I use this particular piece of knowledge?”, while *mathematical modelling* concentrates on the opposite direction, from reality to mathematics and falls under the question: “Where can I find some mathematics to help me with this problem?” (Stillman et al., 2007, p. 689).

Applying mathematics means to relate the real world with mathematics, either by model building, or by simply interplaying. According to Blum and Niss (1991,

p. 37ff), a *problem* is a certain situation involving one or more open questions whose solution presupposes an intellectual challenge, since no direct methods of settling them are available. The term *situation* refers to that part of an individual’s – in this case the student’s – world in which a certain problem is embedded. Model building or simply *modelling* denotes the whole process starting from a real problem and leading to a mathematical model. The original situation is simplified and structured, according to the assumptions and purposes of the solver, and leads to a *real model*. *Mathematising* then comes into play, being the process addressing the transition from the real model to mathematics. This results in a *mathematical model* of the initial situation. All these process steps demand a certain amount of creativity and are therefore in line with Pollak’s (1969) view that formulating a suitable problem in a situation outside mathematics is a creative activity much like discovering mathematics itself.

The modelling cycle is the form in which phases of the modelling processes are depicted and most easily referred to. After working out the mathematical model, a mathematical solution is developed, which has then to be interpreted in order to arrive at a meaning in terms of the real-world situation from which it started. Afterwards, the model has to be validated, and the result is either accepted, if appropriate to the solver’s requirements, or the cycle has to be repeated if the model does not yet satisfy the purposes for which it was created. This arrangement of steps initially stemmed from the work of researchers such as Steiner (1976), Pollak (1979) and Blum (1985). Since then, the previously mentioned steps have constantly been refined and still represent an actual subject of investigation. The starting point of modelling is always a real situation, which gets simplified, idealised and structured, mainly on account of its complexity. For this work, a central didactical position has been adopted from the concept of Blum and Leiß (2007), who gives a detailed representation of the modelling process (see Figure 1).

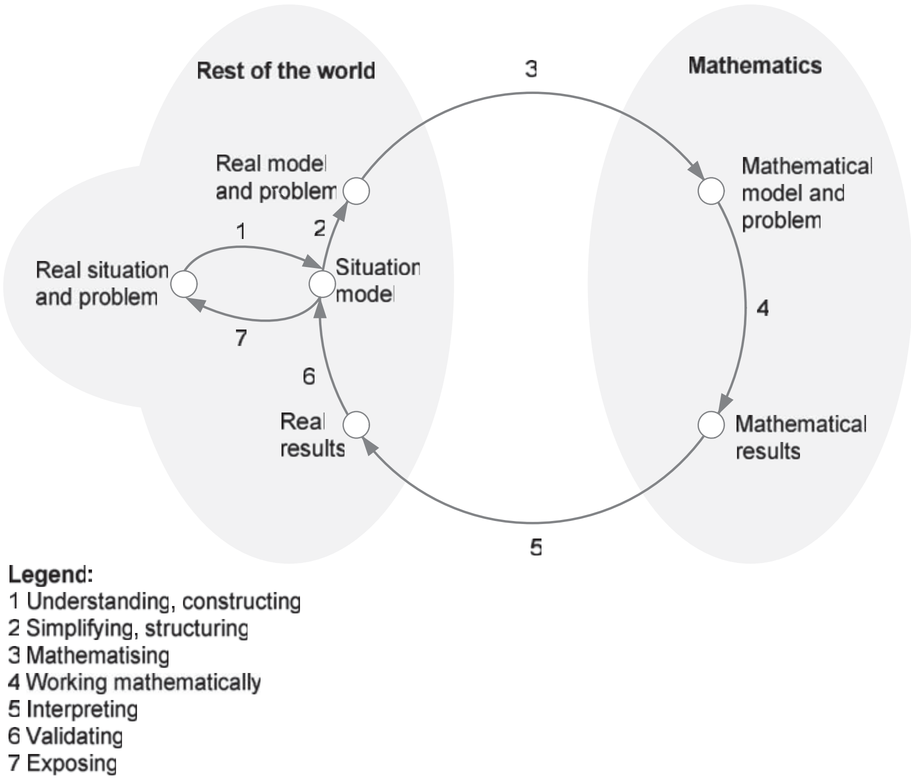

Figure 1: Modelling process (Blum & Leiß, 2007, p. 225)

Applications and modelling in mathematics education may, according to Blum et al. (2002, p. 157), be approached from various perspectives. The fundamental perspective is that of teaching and learning practice, focusing on what should happen in everyday classroom situations at given educational levels. A second perspective is the development and design of curricula, teaching and learning materials or activities, with the aim of setting out plans and conditions for future teaching and learning. The perspective of research is the third, in which research questions may generate as yet undiscovered answers. A final perspective is that of policy, developing instruments, strategies and policies to bring applications and modelling cases into research or practice in an effective manner. The four perspectives could be reformulated in terms of particular professional roles: the role of the teacher or student, the role of curriculum developer, the role of researcher and the role of the decision maker.

In terms of the objectives and scope of this study, we will deal with the first and third perspectives, while the second is dealt with only indirectly, since curriculum elaboration is not the object of the present study.

1.2 Study context

1.2.1 Mathematical character

A particularity of this study is that students are offered a different kind of problems to work on, and with it another type of mathematical environment, than they have typically been accustomed to in mathematics classes. As is widely known by anyone in touch with curricula in grammar schools (*Gymnasien*), and is borne out by students' statements, school textbooks have certain traits that are easily recognised by teachers and students. We found that as soon as they are confronted with a task that is not in line with this conventional format, students may react spontaneously, either becoming unpleasantly surprised and confused, or responding to the challenge positively and becoming even more motivated to perform well. Two statements exemplify the first type of attitude: "I think this is a strange task. How does one tackle it... One doesn't know exactly how to approach this task." (Ema, 8th class), or "I like the kind of task where it is known how to proceed" (Lena, 8th class). The second – and of course more desirable – type of attitude, – is expressed in the statement: "I find really interesting and worthwhile doing this kind of tests" (Markus, 6th class). The students referred to the type of mathematical task in which no calculation is expressly asked for as "tasks whose formulations contain no numbers". They had not been confronted with such tasks before.

Secondly, but still related to the task formulation, it is not clear for students, at least not at first sight, that the problems they are supposed to work on belong to mathematics. Of course, this holds particularly true when no one whom the students already know to be a mathematics teacher is involved. Or, to formulate it more generally, students are not given any clue that they are supposed to be doing something with mathematics, as nothing about the circumstances under which they are working on this specific kind of task hints at mathematics. Students did not identify the given tasks as belonging without any doubt to mathematics, since they were not aware of the target with respect to the tasks. Moreover, when students were asked to draw on their background knowledge when doing mathematical modelling within the investigations of this study, they often mentioned disciplines such as geography, biology and natural sciences, only referring to mathematics in the last instance.

1.2.2 Types of modelling

The main categories for a classification of the existing approaches, comprising the essential features and trends in the modelling literature, are given by Kaiser et al. (2007, p. 2036). The central perspectives adopted from this study are as follows: *realistic or applied modelling,* having as its basic aims pragmatic-utilitarian goals such as solving real-world problems, understanding the real world and promoting of modelling competencies; *contextual modelling,* focusing on subject-related and psychological goals, that is, solving real-world problems; *educational modelling*, with its branches *didactical* and *conceptual modelling*, where the pedagogical and subject-related goals are structuring and promoting the learning process, respectively concept introduction and development; *socio-critical modelling* and *epistemological* or *theoretical modelling*, with theory-oriented goals such as the promotion and development of theory. A special category, described as meta-perspective, is represented by *cognitive modelling*, where psychological goals are meant to be reached.

The analysis of cognitive processes takes place during modelling processes and their understanding; mathematical thinking processes are promoted “by using models as mental images or even physical pictures, or by emphasising modelling as mental process, such as abstraction or generalisation" (Kaiser et al., 2007, p. 2036). It has to be noted here that the meta-perspective, through the nature of its psychological components, could be found in some of the perspective categories enumerated previously. Given its particularities, the present work does not easily fit into this scheme, especially because the focus is on a single modelling step from the modelling cycle, namely *mathematising*, about which not that much has been written.

A further, more succinct classification of modelling by the same authors (Kaiser et al., 2007, p. 2037) allows distinctions to be made between realistic, contextual, educational, socio-critical and epistemological modelling. Most of the tasks used in modelling and worked out with the students fall under (and define the attributes of) this classification. According to these modelling classes, and depending on the nature of the tasks used in the research, the actual work can be categorised as being a realistic or, better, a contextual and epistemological modelling study.

1.3 Research rationale

De Lange (2003, p. 13) states that "deploying mathematics in sophisticated settings such as modern work-based tasks gives students not only motivation and context, but also a concrete foundation from which they can later abstract and generalize". In the same vein, Freudenthal (1968) claims that students must start from situations that have to be mathematised, in order that mathematics is taught as an useful social tool.

1.3.1 Mathematics and numbers

Mathematical knowledge

> is seen as a body of facts and procedures dealing with quantities, magnitudes, and forms, and the relationships among them; knowing mathematics is seen as having mastered these facts and procedures… mathematics is conceptualised as the "science of patterns" an (almost) empirical discipline closely akin to the

> sciences in its emphasis on pattern-seeking on the basis of empirical evidence (Schoenfeld, 1992, p. 334).

The fear and reticence towards mathematics manifested by students in school is not a new problem (Hayward, 1988; Zaslavsky, 1994; Breen, 2001). Among the reasons leading to difficulties with mathematics are "inadequate schools, poor teaching, inappropriate mathematics programs, and stereotypes about who can and who should do math" (Zaslavsky, p. 1). All these can induce further on blockages in understanding concepts that build on fundamental mathematical abilities that have never been properly acquired (or are even missing altogether). "Math anxiety" or "mathephobia" is "a state of mind that makes it difficult or even impossible" to use mathematical skill which a person already has. These might come together with physical symptoms and can take extreme forms when having to take a test. "[T]he term "math anxiety" is reserved for a condition that is traumatic und debilitating" (Zaslavsky, p. 6). Negative experiences in mathematics have the potential to foster long-term negative attitudes towards mathematics and, if teachers are not careful, some projects may "fatally wound kids because they got the first line wrong" (Stillman, 2006, p. 3). Through the proposed type of tasks in this study, it is intended to diminish the chances for this phenomenon, offering students the possibility of trying out another way to do mathematics.

Mathematics, by its rigorous nature, is seen as an abstract discipline (Gonzales Thompson, 1984, p. 113), and unfortunately it is often the case that students fail to look and search for what mathematics is, i.e., behind mathematical methods, algorithms, and so on. They are too quick to calculate (acting almost spontaneously, going directly for calculations) to apply mathematical algorithms and methods, and forget to design and to develop strategies. Cases where "students may not even attempt problems for which they have no ready method, or may curtail their efforts after only a few minutes without success" (Schoenfeld, 1992, p. 27) are not rare. "Problem solving is not usually seen as a goal in itself, but solving problems is seen as facilitating the achievement of other goals" (Schoenfeld, 1992, p. 13); while students are very much concerned to solve given problems in the sense of finding solutions, they do not think about the given situations as a whole, so that they rarely question the decisions they take while doing mathematics.

The basic reason for all these deficiencies and many other aspects of the same phenomenon (Thijsse, 2002), may be the fact that students are not educated in this spirit, and therefore cannot succeed in something they have not learnt or were not taught to do. As a goal for instruction, a pedagogical imperative would be that mathematics

> should develop students' understanding of important concepts in the appropriate core content… Instruction should be aimed at conceptual understanding rather than at mere mechanical skills, and at developing in students the ability to apply the subject matter they have studied with flexibility and resourcefulness… provide students the opportunity to explore a broad range of problems and problem situations, ranging from exercises to open-ended problems and exploratory situations… develop what might be called a "mathematical point of view" – a predilection to analyze and understand, to perceive structure and structural relationships, to see how things fit together… [and] prepare students to become, as much as possible, independent learners, interpreters, and users of mathematics (Schoenfeld, 1992, p. 345).

1.3.2 The concept of tasks without numbers

In the current study, it is intended to find out more about the learning, through learning how hypotheses and assumptions generated by students are treated in environments with no given numbers, by non-mathematical tasks. The novelty consists in the fact that the problem the students are asked to solve is formulated in such a way that it is not obviously a mathematical one. The choice of tasks is deliberately aimed to prevent students engaging spontaneously in calculations, as they usually do when given a typical mathematical problem. The point is to challenge the students' design strategies for problem solving, their meta-cognitive activities, their epistemic actions, so that they find by themselves the mathematical context, methods and means, and thus take responsibility for the entire process, starting from the real situation and ending with the solution of the problem raised by that situation. This is sustained by the fact that

> becoming a good mathematical problem solver – becoming a good thinker in any domain – may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or knowledge (Schoenfeld, 1992, p. 340).

The term "tasks without numbers" is assigned in this study to the specific type of tasks especially designed to be mathematised by the students. The meaning behind the locution "without numbers" is that the initial formulation contains no numbers. There is no explicit requirement for certain calculations, but that does not exclude the possibility that later steps on the chosen way of solving may involve numbers. If students find it appropriate to mathematise in such a manner that numbers fit in and help, even only for exemplifying, then it is totally legitimate that numbers come in. In other words, we assume that the task is "without numbers" in the form we give it at the very beginning, but what happens later on, the phases the task eventually goes through, could easily turn out to be mathematical. In fact, it is hoped that in some later stage of solving, mathematics will come in, in whatever form and by whatever means.

With respect to the mathematical modelling field, to which it belongs, the current study is primarily concentrated on the process of mathematisation and how assumptions and hypotheses come into play in "tasks without numbers". By this it is meant the process within modelling, when students decide on a mathematical approach in order to solve a real-life situation (real in the sense of imaginable for students, not necessarily met in daily life). This process turns out not to be linear at all (Barbosa, 2010; Borromeo Ferri, 2006; Haines & Crouch, 2010), in time or space, where by space we understand the specific context offered by the respective task. It can occur now and then during the modelling process; it can stop at some moment, and reappear later, so that multiple transitions between mathematics and reality can be observed in the solving process.

1.3.3 Research questions

The perspective of this study can be qualified as contextual modelling/modelling eliciting, under an epistemological perspective (Freudenthal, 1973). It is an epistemological issue, because what happens in students' minds when taking different decisions while modelling and, in particular, while mathematising, is puzzling, and therefore demanding to understand and analyse. It is a modelling eliciting perspective as well, due to the "thought-revealing" activities (Lesh et al., 2000) promoted and for engaging mathematical models in order to solve more complex problems, and therefore enhance a better understanding of students' thinking.

The leading and generic research question of the study is:

- *RQ: What is typical for the work of the absolute beginners and students with little experience in mathematical modelling when solving tasks without numbers*?

This affair is rather ample and can be dissected into many other questions, intended to help to answer, and reveal components of the general question. In looking at how students work with assumptions and hypotheses in modelling, the sub-questions of this complex mathematising issue are:

- *RQ_1: Which specific steps of modelling processes are more likely to emerge in tasks without numbers?*
- *RQ_2: How is the need to take certain decisions, to approach particular methods, activated in this type of tasks?*
- *RQ_3: Where do hypotheses and assumptions emerge while modelling by tasks without numbers?*

- *RQ_4: How do assumptions and hypotheses relate to students' epistemic actions by modelling?*
- *RQ_5: What correspondence could be built between assumptions and hypotheses and modelling acquirements?*

Note that when mentioning modelling in the above research questions, we preponderantly mean mathematising as step within modelling. There is no common, detailed study into understanding mathematisation as a process, and therefore no "grating" to allow an eventual formal assessment of mathematising. Through its consistency, the mathematisation process can be analysed and discussed in a more or less subjective manner, since the topic is situated within human science. Nevertheless, mathematics is an exact science, and what it is intended here is a study of how this science is meant to be used and brought into action by students when solving a real problem, with the emphasis on the switch between reality and mathematics. This means that, even if the pedagogical and psychological characteristics of the study imply a degree of uncertainty, mathematics should facilitate the establishment of some clear rules for evaluation.

In order to answer the stated research questions, it is necessary to plan the research activities carefully. Therefore, it is important to make use of the diversity of existing theories in mathematics education research in a structured way, to be able to answer all relevant aspects of research and theory adequately. The research design of this study can be embedded in the following structure, consisting of five aspects of research and theory: aims, methods, questions, situations, and objects (Bikner-Ahsbahs & Prediger, 2006). In the following, we briefly describe these aspects in the context of this dissertation.

The goal is to acquire scientific insights about mathematisation processes in non-mathematical contexts. With respect to the methods used, observations of group discussions on problems in a-priori non-mathematical contexts build the starting point here. Diverse methods intended to provide answers to the research questions are considered, and will be presented at a later stage. The investigation of assumptions and hypotheses within modelling, especially mathematising, is performed in this study regarding epistemic actions of the students. Therefore the local conceptual framework that integrates the analysis of hypotheses and assumptions within the mathematical modelling framework consists of epistemic actions (Hershkowitz et al., 2006, 2007) and assumptions types (Galbraith, 1996; Galbraith & Stillman, 2001), as well as the Scientific Discovery as Dual Search (SDDS) model, which addresses the process of cognition. The research object is represented by modelling tasks without an obvious mathematical nature. As so far as mathematisation is seen as an important part of the mathematical modelling process, it is desirable to learn more about the mathematisation processes in the specified conditions. The experimental part of this study was designed as peer groups, since collaboration in a learning setting has a positive effect on students' learning outcome.

A typology of assumptions in the mathematical modelling process that goes further than the research studies by Galbraith and Stillman (2001), Seino (2005) and Ikahata (2007) is intended to be part of the output of the present study. Given the role that assumptions play in modelling, a micro-analysis and a fine-grained extended categorisation of them seem to be worthwhile. Within the same modelling context, in order to arrive at a better understanding of mathematising, the present study establishes how the way hypotheses are generated influences the decisions students take. Different types of hypotheses will be distinguished, mainly according to the epistemic actions involved. To the author's knowledge, there is no similar study so far, in which the epistemic actions in assumptions and hypotheses are investigated within the modelling framework.

1.4 Dissertation structure

Having presented the context and rationale of this study, a brief summary of its further structuring is given in the following.

The second chapter, Theoretical Background, will describe the role of mathematisation within the modelling cycle. The fundamental mathematical ideas are indicators through which certain actions of students in this study are directly aligned to predefined mathematical structures, attributes, behaviours, etc. The epistemic actions with their RBC constituents, the SDDS model, and the assumptions and hypotheses will be presented afterwards as part of the local conceptual framework, which enhances the processing of empirical data gathered in this research.

The third chapter will point out the design and methodological approach, emphasising particularities of the tasks used, the sampling chosen, as well as the way the video recordings, transcripts and interpretation of the data were employed. The fourth chapter reports in detail about the empirical analysis. Every task is presented, together with the written results from students' transcripts, with three teams of students for each task. Results gathered by every task, and identified as assumptions and hypotheses uttered by the students, will be referred to the SDDS model, and the extent to which the empirical results conform to the original model will be discussed.

In the fifth chapter the assumptions emerging from students' discussions are closely documented and investigated in connection with the modelling stages and towards the RBC model. A categorisation of the assumptions is finally made, based on previous studied types from the research of Galbraith and Stillman (2001). A similar approach is chosen in the sixth chapter, where hypotheses are correlated to the modelling stages, viewed concomitantly with the RBC epistemic actions, and then three types are configured.

In the seventh chapter a generic outlook on the students' achievements is provided, and its correspondent impacts for the main issues of the study, i.e., modelling and in particular mathematising, are reviewed. Based on students' level of stating hypotheses and assumptions, as well as their frequency during validation within the modelling process, modelling acquirements levels are proposed.